6.

Systems of linear equations

- Systems of equations
- A systematic algebraic approach for solving systems of linear equations using a matrix form of presentation (and otherwise)
- Calculators and row echelon form
- Systems of linear equations having no solution
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One of the examples of the previous chapter involved setting up a 'system' of three equations in three unknowns and then using a calculator to 'solve the system':

$$\begin{aligned} 9 &= 2\lambda + 3\mu + \eta \\ 5 &= \lambda - 2\mu - 2\eta \\ 2 &= -\lambda + 4\mu + \eta \end{aligned}$$

$$\begin{cases} 9 = 2\lambda + 3\mu + \eta \\ 5 = \lambda - 2\mu - 2\eta \\ 2 = -\lambda + 4\mu + \eta \end{cases} \Bigg| \lambda, \mu, \eta$$

$$\{\lambda = 3, \mu = 2, \eta = -3\}$$

Attempt the next two situations, which again each involve setting up and then solving, a system of three equations in three unknowns. However, in each case try to solve the system *without* using the equation solving ability of your calculator.

Situation One

A total of \$15 000 is invested in three different investment schemes, A, B and C, for two years. At the end of the first year the total value of the investment package is \$15 430. At the end of the second year the total value is \$16 317. The performance of each scheme in each year is shown below.

Scheme	Initial		After 1 year		After 2 years
A	$\$x$	10% increase →		5% increase →	
B	$\$y$	5% increase →		10% increase →	
C	$\$z$	5% decrease →		5% increase →	
Totals	\$15 000	→	\$15 430	→	\$16 317

Write three equations involving x, y and z and hence determine the initial investment made into each of the schemes.

Situation Two

A dog food manufacturer makes three types of dog food mix. Each type is sold in bags containing 5 kg with the ratio of meat : rice : vegetables being as follows:

Mix A	*Meaty Lumps*	5 : 3 : 2
Mix B	*Balanced Lumps*	1 : 1 : 3
Mix C	*Vegetarian Lumps*	0 : 3 : 2

The manufacturer orders 3350 kg of meat, 4850 kg of vegetables and 4300 kg of rice for a particular production run. The run involves no weight loss for any of the ingredients and all the quantities ordered are exactly used up. If the run produces x bags of Mix A, y bags of Mix B and z bags of Mix C write three equations that apply and solve them to find x, y and z.

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WS
Systems of equations

Systems of equations

Did you manage to solve the systems of linear equations obtained from the previous situations by applying the elimination technique you are probably familiar with as a means of solving two equations in two unknowns? In this method we use the equations 'against each other' to eliminate one of the variables.

For example, for a system involving two linear equations:

$$\begin{aligned} x + y &= 11 && [1] \\ 3x + 2y &= 26 && [2] \end{aligned}$$

Equation [1] $\times$ 3	$3x + 3y = 33$	
Equation [2]	$3x + 2y = 26$	
Subtraction eliminates x:	$y = 7$	
Substitution back into Equation [1] gives	$x + 7 = 11$	
$\therefore$	$x = 4$	

This same elimination technique can be used for 3 equations involving three unknowns, as shown below.

Consider the three equations:
$$\begin{cases} x + 2y - 3z = 9 & [1] \\ 2x - y + z = 0 & [2] \\ -3x + 4y - 2z = 12 & [3] \end{cases}$$

Equation [1] $\times$ 2	$2x + 4y - 6z = 18$	
Equation [2]	$2x - y + z = 0$	
Subtraction eliminates x:	$5y - 7z = 18$	[4]
Equation [1] $\times$ 3	$3x + 6y - 9z = 27$	
Equation [3]	$-3x + 4y - 2z = 12$	
Addition eliminates x:	$10y - 11z = 39$	[5]
Equation [4] $\times$ 2	$10y - 14z = 36$	
Equation [5]	$10y - 11z = 39$	
Subtraction eliminates y:	$-3z = -3$	
$\therefore$	$z = 1$	
Substitution back into equation [4] gives	$y = 5$	
Substitution back into equation [1] gives	$x = 2$	

Thus $x = 2$, $y = 5$ and $z = 1$.

Thus by using the equations 'against each other' to systematically eliminate variables until we are left with one equation in one unknown, systems of three equations involving three unknowns can be solved, provided of course that a solution exists.

Alternatively we could use the ability of some calculators to solve systems of equations like this.

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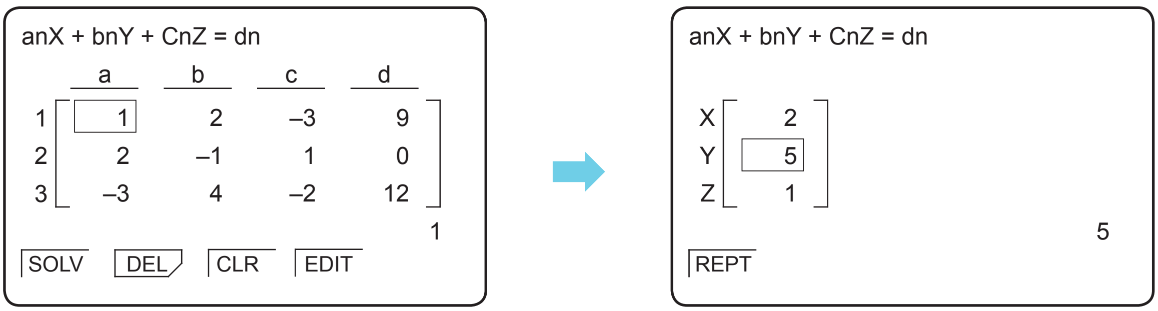
A systematic algebraic approach for solving systems of linear equations using a matrix form of presentation (and otherwise)

Whilst many calculators allow us to input equations in the same way as we write them, this is not the case for all calculators. The display below left shows how one calculator requires the system of equations

$$\begin{cases} x + 2y - 3z = 9 \\ 2x - y + z = 0 \\ -3x + 4y - 2z = 12 \end{cases}$$

to be input. The solution to the system can then be displayed, as shown below right.



Notice that this calculator only requires us to enter the *coefficients* of the variables, and the constant term, of each equation, e.g. 1, 2, –3 and 9 for the equation $x + 2y - 3z = 9$.

Our method of manipulating the equations to eliminate variables could also be set out in this way:

$$\text{Writing} \quad \begin{cases} x + 2y - 3z = 9 \\ 2x - y + z = 0 \\ -3x + 4y - 2z = 12 \end{cases} \quad \text{as} \quad \begin{bmatrix} 1 & 2 & -3 & 9 \\ 2 & -1 & 1 & 0 \\ -3 & 4 & -2 & 12 \end{bmatrix} \begin{array}{l} \leftarrow \text{Row 1 } (r_1) \\ \leftarrow \text{Row 2 } (r_2) \\ \leftarrow \text{Row 3 } (r_3) \end{array}$$

We then manipulate these rows in the same way as we would manipulate the equations. This means we can

- multiply any row by a number,
- add or subtract rows,
- interchange one row with another.

$$\begin{array}{l} r_1 \\ r_2 \\ r_3 \end{array} \quad \begin{bmatrix} 1 & 2 & -3 & 9 \\ 2 & -1 & 1 & 0 \\ -3 & 4 & -2 & 12 \end{bmatrix}$$

$$\begin{array}{l} r_1 \\ r_2 - 2r_1 \\ r_3 + 3r_1 \end{array} \quad \begin{bmatrix} 1 & 2 & -3 & 9 \\ 0 & -5 & 7 & -18 \\ 0 & 10 & -11 & 39 \end{bmatrix} \begin{array}{l} \\ \leftarrow \text{new } (r_2) \\ \leftarrow \text{new } (r_3) \end{array}$$

$$\begin{array}{l} r_1 \\ r_2 \\ r_3 + 2r_2 \end{array} \quad \begin{bmatrix} 1 & 2 & -3 & 9 \\ 0 & -5 & 7 & -18 \\ 0 & 0 & 3 & 3 \end{bmatrix} \begin{array}{l} \\ \\ \leftarrow \text{new } (r_3) \end{array}$$

Row 3 now tells us that	$3z = 3$,	thus	$z = 1$.
Using this in row 2	$-5y + 7 = -18$,	thus	$y = 5$.
And from row 1	$x + 10 - 3 = 9$,	thus	$x = 2$.

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Note

- For the system $\begin{cases} x + 2y - 3z = 9 \\ 2x - y + z = 0 \\ -3x + 4y - 2z = 12 \end{cases}$ the **matrix of coefficients**, or **coefficient matrix** is $\begin{bmatrix} 1 & 2 & -3 \\ 2 & -1 & 1 \\ -3 & 4 & -2 \end{bmatrix}$

  This coefficient matrix has 3 rows and 3 columns.

  From the coverage of Matrices in *Mathematics Specialist Unit Two* the reader should be aware that the above system of equations could be expressed in 'matrix form', and the inverse of the *coefficient matrix* used to determine a solution (as the *Preliminary Work* section at the beginning of this book reminded us). However, that is not the method explored here.

- If we enlarge the matrix to include the right hand side of each equation we have the **augmented matrix**: $\begin{bmatrix} 1 & 2 & -3 & 9 \\ 2 & -1 & 1 & 0 \\ -3 & 4 & -2 & 12 \end{bmatrix}$

  (The word *augment* means to increase or enlarge.)
  This augmented matrix has 3 rows and 4 columns.

- The augmented matrix is sometimes shown with a vertical line separating the two parts of the matrix, as shown on the right. $\left[\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 2 & -1 & 1 & 0 \\ -3 & 4 & -2 & 12 \end{array}\right]$

- To solve the system of equations we manipulate the rows until the coefficient matrix shows only zeros below the leading diagonal.

  i.e. until the augmented matrix is of the form: $\left[\begin{array}{ccc|c} \# & \# & \# & \# \\ 0 & \# & \# & \# \\ 0 & 0 & \# & \# \end{array}\right]$

  Notice the zeros below the 'dotted line step formation' in the above matrix. In this form the augmented matrix is said to be in **row echelon** form, the word *echelon* coming from its military use for a battle formation sometimes used by soldiers or warships. (A matrix is in echelon form when any rows in which all elements are zero occur at the bottom of the matrix and, as we move down the other rows, the first non zero elements in successive rows move right.)

  The following matrices are all in echelon form:

$$\begin{bmatrix} 3 & 5 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 3 & -4 & 1 \\ 0 & 2 & -3 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 & -3 \\ 0 & 2 & -3 & 5 \\ 0 & 0 & 2 & -10 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 & -3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- By interpreting the operations we might carry out in the elimination method, as operations on the rows of the augmented matrix, we obtain the following **elementary row operations** that we can use:

  – Interchanging rows.
  – Multiplying a row by a non-zero constant.
  – Adding a multiple of one row to a multiple of another row.

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The following steps could be followed to systematically reduce the augmented matrix to echelon form for a system of three equations in three unknowns:

1. Combine suitable multiples of first and second rows to make the first element of the second row zero.
2. Combine suitable multiples of first and third rows to make the first element of the third row zero.
3. Use new 2nd and 3rd rows to reduce 2nd element of row 3 to zero.

   (Steps 1 and 2 can be easier if row 1 starts with a 1. Thus if some other row starts with a 1 move this row to row 1.)

The examples that follow run through the procedures involved using the matrix form of presentation. However, at the time of writing, the syllabus for this unit requires you to be able to *use elementary techniques of elimination to solve systems of linear equations*. Whether you use the augmented matrix approach, or you manipulate the equations themselves, is not stipulated. Thus whilst the augmented matrix approach is not specifically mentioned in the syllabus it is included here as a possible style of presentation for solving systems of linear equations. Adopt it if you wish.

Calculators and row echelon form

Some calculators are able to reduce a matrix to echelon form. Indeed some are able to reduce a matrix beyond the *row echelon form* shown on the right to the *reduced row echelon form* shown below it.

$$\left[\begin{array}{ccc|c} \# & \# & \# & \# \\ 0 & \# & \# & \# \\ 0 & 0 & \# & \# \end{array}\right]$$

This second form makes the determination of the solutions just a matter of reading the final column.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \# \\ 0 & 1 & 0 & \# \\ 0 & 0 & 1 & \# \end{array}\right]$$

Some calculator programs (and internet sites) reduce an augmented matrix to echelon form *displaying each row operation performed as it goes*! However these steps may not be the same steps you would choose to follow if doing the row operations yourself because the calculator may follow a certain procedure everytime and might not be programmed to see 'shortcuts' that could exist.

You are certainly encouraged to explore the capability of your calculator and the internet in this regard but also make sure that you can demonstrate your own ability to solve systems of equations systematically without such assistance.

Note
- The process of reducing the augmented matrix to echelon form, and the similar process of manipulating the equations to eliminate variables, is called **Gaussian elimination**.
- Because there are often different ways of combining the elementary row operations to reduce an augmented matrix to row echelon form, this echelon form is not unique. However the reduced form shown above is unique.

EXAMPLE 1 **Solution using augmented matrix**

Solve the system of linear equations , shown right.

$$\begin{aligned} 2x + y - 2z &= 11 \\ x + 2y - 3z &= 17 \\ -3x - y + 4z &= -21 \end{aligned}$$

Solution

First write the Augmented matrix:

$$\begin{array}{l} r_1 \\ r_2 \\ r_3 \end{array} \quad \begin{bmatrix} 2 & 1 & -2 & 11 \\ 1 & 2 & -3 & 17 \\ -3 & -1 & 4 & -21 \end{bmatrix}$$

Switch rows 1 and 2 to make first element of first row a 1:

$$\begin{array}{l} r_2 \\ r_1 \\ r_3 \end{array} \quad \begin{bmatrix} 1 & 2 & -3 & 17 \\ 2 & 1 & -2 & 11 \\ -3 & -1 & 4 & -21 \end{bmatrix} \quad \begin{array}{l} \leftarrow \text{new } r_1 \\ \leftarrow \text{new } r_2 \\ \end{array}$$

Combine suitable multiples of first and second rows to make the first element of the second row zero (and similarly use the first and third rows to make the first element of the third row a zero).

$$\begin{array}{l} r_1 \\ r_2 - 2r_1 \\ r_3 + 3r_1 \end{array} \quad \begin{bmatrix} 1 & 2 & -3 & 17 \\ 0 & -3 & 4 & -23 \\ 0 & 5 & -5 & 30 \end{bmatrix} \quad \begin{array}{l} \\ \leftarrow \text{new } r_2 \\ \leftarrow \text{new } r_3 \end{array}$$

Use new 2nd and 3rd rows to reduce 2nd element of row 3 to zero.

$$\begin{array}{l} r_1 \\ r_2 \\ 3r_3 + 5r_2 \end{array} \quad \begin{bmatrix} 1 & 2 & -3 & 17 \\ 0 & -3 & 4 & -23 \\ 0 & 0 & 5 & -25 \end{bmatrix} \quad \begin{array}{l} \\ \\ \leftarrow \text{new } r_3 \end{array}$$

Row 3 now tells us that	$5z = -25$,	thus	$z = -5$.
Using this in row 2	$-3y - 20 = -23$,	thus	$y = 1$.
And from row 1	$x + 2 + 15 = 17$,	thus	$x = 0$.

Thus $x = 0$, $y = 1$ and $z = -5$.

Alternatively, if you prefer to deal with the equations themselves, rather than manipulate the augmented matrix, you could follow similar steps to arrive at the same solution. This alternative form of presentation is shown on the next page as 'Example 1 repeated'. In the examples that appear later in this chapter each example is shown solved using an augmented matrix approach and then repeated without using the augmented matrix. Each form of presentation involves the systematic elimination of variables until one equation in one unknown is reached.

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EXAMPLE 1 REPEATED **Solution without using augmented matrix**

Solve the system of linear equations, shown right.

$$\begin{aligned} 2x + y - 2z &= 11 \\ x + 2y - 3z &= 17 \\ -3x - y + 4z &= -21 \end{aligned}$$

Solution

Given the three equations:

$$\begin{cases} 2x + y - 2z = 11 & [1] \\ x + 2y - 3z = 17 & [2] \\ 3x + y + 4z = -21 & [3] \end{cases}$$

Use equations [1] and [2] to eliminate x.

Equation [1]	$2x + y - 2z = 11$	
Equation [2] $\times$ 2	$2x + 4y - 6z = 34$	
Subtraction eliminates x:	$-3y + 4z = -23$	[4]

Use equations [2] and [3] to eliminate x.

Equation [2] $\times$ 3	$3x + 6y - 9z = 51$	
Equation [3]	$-3x - y + 4z = -21$	
Addition eliminates x:	$5y - 5z = 30$	[5]

Equation [4] $\times$ 5	$-15y + 20z = -115$
Equation [5] $\times$ 3	$15y - 15z = 90$
Addition eliminates y:	$5z = -25$
$\therefore$	$z = -5$
Substitution back into equation [4] gives	$y = 1$
Substitution back into equation [1] gives	$x = 0$

Thus $x = 0$, $y = 1$ and $z = -5$, as before.

EXAMPLE 2 **Augmented matrix approach**

Solve the system of linear equations, shown right.

$$\begin{aligned} 2x + y + z &= 1 \\ 2y + z &= 1 \\ 3x + y - z &= 8 \end{aligned}$$

Solution

First write the Augmented matrix:

$$\begin{array}{l} r_1 \\ r_2 \\ r_3 \end{array} \quad \begin{bmatrix} 2 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 3 & 1 & -1 & 8 \end{bmatrix}$$

$$\begin{array}{l} r_1 \\ r_2 \\ 2r_3 - 3r_1 \end{array} \quad \begin{bmatrix} 2 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & -1 & -5 & 13 \end{bmatrix} \quad \begin{array}{l} \\ \\ \leftarrow \text{new } r_3 \end{array}$$

$$\begin{array}{l} r_1 \\ r_2 \\ 2r_3 + r_2 \end{array} \quad \begin{bmatrix} 2 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & -9 & 27 \end{bmatrix} \quad \begin{array}{l} \\ \\ \leftarrow \text{new } r_3 \end{array}$$

From which we can determine that $z = -3$, $y = 2$ and $x = 1$.

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EXAMPLE 2 REPEATED **Solution without using augmented matrix**

Solve the following system of linear equations.

$$\begin{aligned} 2x + \ y + z &= 1 \\ 2y + z &= 1 \\ 3x + \ y - z &= 8 \end{aligned}$$

Solution

Given the three equations:

$$\begin{cases} 2x + y + z = 1 & [1] \\ 2y + z = 1 & [2] \\ 3x + y - z = 8 & [3] \end{cases}$$

Use equations [1] and [3] to eliminate x.

Equation [1] $\times$ 3	$6x + 3y + 3z = 3$	
Equation [3] $\times$ 2	$6x + 2y - 2z = 16$	
Subtraction eliminates x:	$y + 5z = -13$	[4]

Use equations [2] and [4] to eliminate x.

Equation [2]	$2y + \ z = 1$
Equation [4] $\times$ 2	$2y + 10z = -26$
Subtraction eliminates y:	$-9z = 27$
$\therefore$	$z = -3$
Substitution back into equation [4] gives	$y = 2$
Substitution back into equation [1] gives	$x = 1$

Thus $z = -3$, $y = 2$ and $x = 1$, as before.

It should be remembered that the purpose of reducing the augmented matrix to echelon form is to give one row in our matrix in which two of the variables have been eliminated. In some cases the numbers involved may enable this situation to be obtained quickly without reducing to echelon form. For example, consider the following augmented matrix:

$$\begin{array}{l} r_1 \\ r_2 \\ r_3 \end{array} \quad \begin{bmatrix} 1 & 2 & 3 & 13 \\ 2 & 5 & 9 & 33 \\ 0 & 2 & -3 & -4 \end{bmatrix}$$

$$\begin{array}{l} r_1 \\ r_2 - 2r_1 \\ r_3 \end{array} \quad \begin{bmatrix} 1 & 2 & 3 & 13 \\ 0 & 1 & 3 & 7 \\ 0 & 2 & -3 & -4 \end{bmatrix} \quad \leftarrow \text{new } r_2$$

Rather than aim for echelon form we could use 'new $r_3 = r_3 + r_2$' to produce a second zero in a row of the coefficient matrix.

$$\begin{array}{l} r_1 \\ r_2 \\ r_3 + r_2 \end{array} \quad \begin{bmatrix} 1 & 2 & 3 & 13 \\ 0 & 1 & 3 & 7 \\ 0 & 3 & 0 & 3 \end{bmatrix} \quad \leftarrow \text{new } r_3$$

Thus $3y = 3$ and hence $y = 1$, $z = 2$ and $x = 5$.

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Exercise 6A

Each of the following shows an augmented matrix that has been reduced to echelon form by elementary row operations. In each case the original augmented matrix was for a system of 3 equations with the coefficients of the three variables x, y and z forming the first three columns respectively. Find x, y and z in each case.

1 $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 5 \end{array}\right]$

2 $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -2 \end{array}\right]$

3 $\left[\begin{array}{ccc|c} 2 & 1 & 1 & 4 \\ 0 & 2 & 5 & 15 \\ 0 & 0 & 1 & 1 \end{array}\right]$

4 $\left[\begin{array}{ccc|c} 1 & 0 & -2 & 7 \\ 0 & 3 & 2 & 6 \\ 0 & 0 & -1 & 3 \end{array}\right]$

5 $\left[\begin{array}{ccc|c} 1 & 3 & 2 & 27 \\ 0 & 5 & -3 & 2 \\ 0 & 0 & 2 & 12 \end{array}\right]$

6 $\left[\begin{array}{ccc|c} 2 & 1 & 1 & -3 \\ 0 & 3 & -2 & 0 \\ 0 & 0 & -3 & 9 \end{array}\right]$

Write the augmented matrix for each of the following systems of equations.

7 $$\begin{aligned} 3x + 2y &= 10 \\ x - 4y &= 8 \end{aligned}$$

8 $$\begin{aligned} -x + 5y &= 12 \\ 2x + 3y &= 2 \end{aligned}$$

9 $$\begin{aligned} x + 4y + 3z &= 18 \\ 3x + y + 2z &= 11 \\ 5x + 2y + z &= 12 \end{aligned}$$

10 $$\begin{aligned} 2x + 3z &= 14 \\ 4x + y - z &= 0 \\ 2x + y + 6z &= 26 \end{aligned}$$

11 $$\begin{aligned} 3x + 2y &= 8 \\ x + 2z &= 8 \\ 2y - z &= -1 \end{aligned}$$

12 $$\begin{aligned} x + 3y - 5z &= 2 \\ 2x + y + 7z &= 37 \\ -x + z &= 3 \end{aligned}$$

Using an augmented matrix approach, or otherwise, solve each of the following systems of equations **without the assistance of the solve facility of a calculator**. Whichever method of presentation you choose, how you are combining rows or equations needs to be clearly stated. I.e. include in your working explanation like $r_2 - 2r_1$, or Equation [1] $\times$ 2, etc.

(If you prefer to manipulate the equations rather than use the augmented matrix approach it is suggested that you at least do some of the questions using the matrix approach.)

13 $$\begin{aligned} x + 3y &= 34 \\ 2x + 5y &= 59 \end{aligned}$$

14 $$\begin{aligned} 2x + 3y &= 4 \\ 4x + 9y &= 2 \end{aligned}$$

15 $$\begin{aligned} x + 2y + z &= 7 \\ y + 3z &= 7 \\ 3x + 3y + z &= 14 \end{aligned}$$

16 $$\begin{aligned} x + y + z &= 6 \\ x + 2y + 4z &= 6 \\ 2x + 3y - 3z &= 20 \end{aligned}$$

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17
$$\begin{aligned} x + 4z &= -1 \\ 2x + y + 3z &= 8 \\ 5x + y &= 35 \end{aligned}$$

18
$$\begin{aligned} x + 2y - z &= 3 \\ 2x + 3y + 2z &= -1 \\ 3x + 7y - 2z &= 6 \end{aligned}$$

19
$$\begin{aligned} 2x + y &= 11 \\ x + 2y - z &= 15 \\ 3x + 9y + z &= 16 \end{aligned}$$

20
$$\begin{aligned} 2x + 4y - 3z &= 1 \\ 2x + 5y - 2z &= 5 \\ 3x + 7y - 3z &= 7 \end{aligned}$$

21
$$\begin{aligned} 3x + 4y + 5z &= 14 \\ 5x + 7y + 6z &= 13 \\ x + y + z &= 3 \end{aligned}$$

22
$$\begin{aligned} 2x + z &= 4 \\ 2x + 3y + 3z &= 3 \\ 5x + y + 3z &= 10 \end{aligned}$$

23
$$\begin{aligned} x + y + 2z &= 6 \\ 3x + 2y + z &= 7 \\ 5x + 4y + 4z &= 19 \end{aligned}$$

24
$$\begin{aligned} w + x - y + 3z &= -1 \\ x + 2y - 3z &= -2 \\ w + 2x + 2y + z &= 0 \\ 2w + 3x + 2y + 7z &= 4 \end{aligned}$$

Solve each of the following without simply using the solve facility on a calculator.

25 A manufacturer of lawn mowers makes two types, the standard and the deluxe. To send 270 standard models and 220 deluxe models overseas, the manufacturer uses two types of container. Each type A container can hold 5 standard models and 2 deluxe models when full, whilst each type B container can hold 3 standard models and 4 deluxe models when full, no other arrangements quite use all the available space. By supposing that the manufacturer uses x type A containers and y type B containers write two equations that apply if the machines are sent with all the containers full. Hence determine x and y.

26 A vet is called to the zoo to treat a large animal. To sedate the animal the vet decides to powder up a number of tablets and give the resulting 'super-tablet' to the animal in its feed. To make this super-tablet the vet uses a combination of three tablets, P, Q and R, using p Ps, q Qs and r Rs.

Each P tablet has	250 mg of X,	10 mg of Y,	and	50 mg of Z.
Each Q tablet has	500 mg of X,	5 mg of Y,	and	100 mg of Z.
Each R tablet has	200 mg of X,	20 mg of Y,	and	100 mg of Z.

The super-tablet the vet makes contains 8 g of X, 470 mg of Y and 2.8 g of Z. Write three equations that apply and hence find how many of each tablet the vet used.

27 When making x kg of fertiliser A, y kg of fertiliser B and z kg of fertiliser C, a company uses 610 kg of compound P, 180 kg of compound Q and 210 kg of compound R.

1 kg of fertiliser A contains	0.5 kg P,	0.1 kg Q,	and	0.4 kg R.
1 kg of fertiliser B contains	0.3 kg P,	0.5 kg Q,	and	0.2 kg R.
1 kg of fertiliser C contains	0.8 kg P,	0.1 kg Q,	and	0.1 kg R.

a Write 3 equations involving x, y and z for the above information.

b Solve the equations to determine x, y and z.

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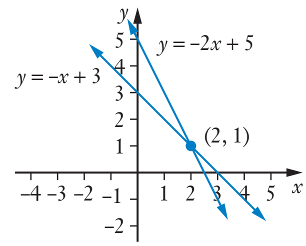
Systems of linear equations having no solution

If we solve two equations in two unknowns, for example:



$$\begin{cases} 2x + y = 5 \\ x + y = 3 \end{cases}$$

the solution gives the coordinates of the point of intersection of the two lines, in this case (2, 1). This is the only solution the system of equations has.

We say that $x = 2$, $y = 1$ is the **unique** solution.

Suppose instead we are asked to solve

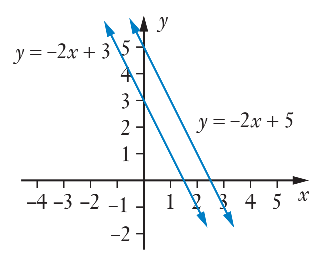
$$\begin{cases} 2x + y = 5 & \text{Equation [1]} \\ 2x + y = 3 & \text{Equation [2]} \end{cases}$$

Equation [1] – Equation [2] gives $\quad 0x + 0y = 2.$

It is not possible to find values for x and y that satisfy this last equation.

The equations $2x + y = 5$ and $2x + y = 3$ contradict each other. We say the equations are **inconsistent**. It is **not** possible to find a solution to this system of equations. This contrasts with the systems met earlier in this chapter which all had unique solutions.

Graphically, $2x + y = 5$ (i.e. $y = -2x + 5$) and $2x + y = 3$ (i.e. $y = -2x + 3$), represent parallel lines. They have no points in common and so the system of equations has no solution.



We would obtain the same conclusion, i.e. that there is no solution, if we use the augmented matrix approach:

$$\begin{array}{l} r_1 \\ r_2 \end{array} \quad \begin{bmatrix} 2 & 1 & 5 \\ 2 & 1 & 3 \end{bmatrix}$$

$$\begin{array}{l} r_1 \\ r_2 - r_1 \end{array} \quad \begin{bmatrix} 2 & 1 & 5 \\ 0 & 0 & -2 \end{bmatrix} \quad \leftarrow \text{new } r_2$$

This last line is telling us that $0x + 0y = -2$ so again we conclude 'no solution'.

Similarly, with a system of three equations involving three unknowns, if our working leads to the claim that $0x + 0y + 0z = a$, for $a \neq 0$, we again conclude 'no solution'.

For example, consider the system:

$$\begin{aligned} x + 3y + z &= 2 \\ x + 2y + 4z &= 0 \\ 2x + 7y - z &= 1 \end{aligned}$$

The augmented matrix is

$$\left[\begin{array}{ccc:c} 1 & 3 & 1 & 2 \\ 1 & 2 & 4 & 0 \\ 2 & 7 & -1 & 1 \end{array}\right]$$

which can be reduced to

$$\left[\begin{array}{ccc:c} 1 & 3 & 1 & 2 \\ 0 & -1 & 3 & -2 \\ 0 & 0 & 0 & -5 \end{array}\right]$$

The last line is claiming that $0x + 0y + 0z = a$, for $a \neq 0$, and hence we conclude 'no solution'. The initial equations are **inconsistent**.

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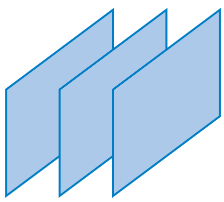
We know from the previous chapter that an equation of the form $ax + by + cz = d$ is the equation of a plane (i.e. a flat surface). Thus, if we have a system of three such equations which has 'no solution' it means that the three equations represent planes that have no point(s) common to all three planes. Two of the planes could be parallel for example. The various possibilities are shown below.

Three parallel planes



Example:

$$\begin{aligned} x - 2y + 3z &= 5 \\ -x + 2y - 3z &= 8 \\ 2x - 4y + 6z &= 7 \end{aligned}$$

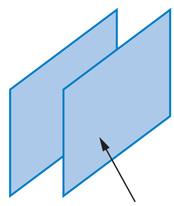
Augmented matrix can be reduced to:

$$\begin{bmatrix} 1 & -2 & 3 & 5 \\ 0 & 0 & 0 & 13 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

$0x + 0y + 0z = 13$!!
$0x + 0y + 0z = -3$!!

No solution.

One plane listed twice and one other parallel plane



Two equations defining the same plane.

Example:

$$\begin{aligned} x - 2y + 3z &= 5 \\ x - 2y + 3z &= 8 \\ 2x - 4y + 6z &= 10 \end{aligned}$$

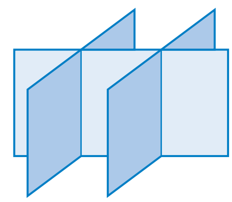
Augmented matrix can be reduced to:

$$\begin{bmatrix} 1 & -2 & 3 & 5 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$0x + 0y + 0z = 3$!!

No solution.

Two parallel planes and one other plane



Example:

$$\begin{aligned} x - 2y + 3z &= 5 \\ x + 4y - 2z &= 3 \\ 2x - 4y + 6z &= 11 \end{aligned}$$

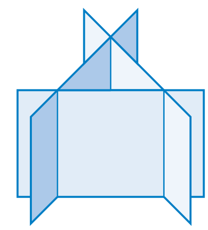
Augmented matrix can be reduced to:

$$\begin{bmatrix} 1 & -2 & 3 & 5 \\ 0 & 6 & -5 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$0x + 0y + 0z = 1$!!

No solution.

Intersecting pairs of planes form parallel lines



Example:

$$\begin{aligned} x + y - z &= 1 \\ x - 2y - z &= -2 \\ -x + 5y + z &= 7 \end{aligned}$$

Augmented matrix can be reduced to:

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$0x + 0y + 0z = 1$!!

No solution.

Note that equations of the form: $ax + by + cz = d$
$\mu(ax + by + cz = e)$, for some constant μ $(\neq 0)$,
define the same plane if $d = e$ and separate parallel planes if $d \neq e$.

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Systems of linear equations having infinitely many solutions

Consider the following system of 2 equations in 2 unknowns: $\begin{cases} 2x + y = 3 \\ 4x + 2y = 6 \end{cases}$

The 2 equations define one line because $4x + 2y = 6$ is $2(2x + y = 3)$.

The coordinates of each and every point on the line $2x + y = 3$ provide a solution to the system.
The system of equations is **insufficient** to give a unique solution and instead has an **infinite number of solutions**.

The augmented matrix for the system is

$$\begin{array}{l} r_1 \\ r_2 \end{array} \quad \begin{bmatrix} 2 & 1 & 3 \\ 4 & 2 & 6 \end{bmatrix}$$

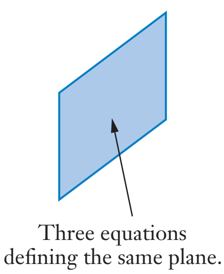
which reduces to

$$\begin{array}{l} r_1 \\ r_2 - 2r_1 \end{array} \quad \begin{bmatrix} 2 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} \\ \leftarrow \text{new } r_2 \end{array}$$

The last line states that $0x + 0y = 0$, which is true for all values of x and y. Solutions come from row 1, $2x + y = 3$, for which there are infinite solutions.

In three dimensions infinite solutions will occur when the three planes meet in a line or a plane as shown below.

The same plane listed three times



Example:

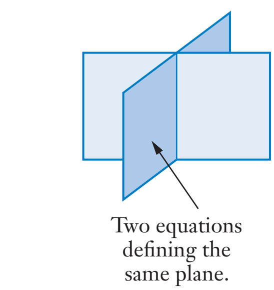
$$\begin{aligned} x - 2y + 3z &= 5 \\ 2x - 4y + 6z &= 10 \\ -x + 2y - 3z &= -5 \end{aligned}$$

Augmented matrix can be reduced to:

$$\begin{bmatrix} 1 & -2 & 3 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$0x + 0y + 0z = 0$ is uninformative.
Solutions from 1st row.
Infinite solutions.

One plane listed twice and one other non-parallel plane



Example:

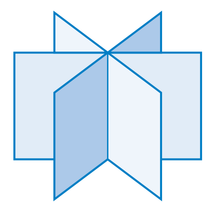
$$\begin{aligned} x - 2y + 3z &= 5 \\ 2x - 4y + 6z &= 10 \\ x + 3y + 3z &= 7 \end{aligned}$$

Augmented matrix can be reduced to:

$$\begin{bmatrix} 1 & -2 & 3 & 5 \\ 0 & 5 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$0x + 0y + 0z = 0$ is uninformative.
Solutions from 1st and 2nd rows.
Infinite solutions.

Three planes meeting in a common line



Example:

$$\begin{aligned} x + y - 2z &= -1 \\ x + 3y + z &= 0 \\ -2x - 4y + z &= 1 \end{aligned}$$

Augmented matrix can be reduced to:

$$\begin{bmatrix} 1 & 1 & -2 & -1 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$0x + 0y + 0z = 0$ is uninformative.
Solutions from 1st and 2nd rows.
Infinite solutions.

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Note: In the last situation, the three planes meeting in a common line, each of the original equations is a linear combination of the other two:

$$(-1)(x + y - 2z = -1) + (-1)(x + 3y + z = 0) = (-2x - 4y + z = 1)$$

This means that any one of the equations is not telling us any more about the three variables than could be obtained from the other two equations. Thus we really only have two pieces of information about the three unknowns and, provided these pieces of information are not contradictory, we have infinite solutions.

However this linear combination is not always obvious.

If the linear combination holds for the x, y and z parts of the equation but not the constant terms, then the 'two pieces of information' we have are contradictory and we have the 'no solution situation' of *intersecting pairs of planes forming three parallel lines* seen earlier.

For example for the system

$$\begin{aligned} x + y - z &= 1 \\ x - 2y - z &= -2 \\ -x + 5y + z &= 7 \end{aligned}$$

given as an example of that situation on an earlier page

$$\begin{aligned} (1)(x + y - z = 1) + (-2)(x - 2y - z = -2) &= (-x + 5y + z = 5) \\ &\neq (-x + 5y + z = 7) \end{aligned}$$

the linear combination differing from the third equation only in the constant term.

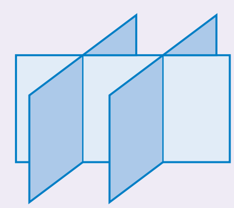
EXAMPLE 3

Determine whether the system shown on the right has a unique solution, no solution or an infinite number of solutions. If there is a unique solution, find it.

$$\begin{aligned} x - 2y + 3z &= 5 \\ -x + 2y - 3z &= 4 \\ 2x + y - z &= 4 \end{aligned}$$

Solution

Noticing that the first two equations define separate parallel planes and the third equation is not parallel to the other two, we have the situation shown on the right. Thus there is no solution.



Alternatively, had the parallel nature of two of the planes escaped our notice, manipulation of the augmented matrix would yield the same conclusion, as shown below.

$$\begin{array}{l} r_1 \\ r_2 \\ r_3 \end{array} \quad \begin{bmatrix} 1 & -2 & 3 & 5 \\ -1 & 2 & -3 & 4 \\ 2 & 1 & -1 & 4 \end{bmatrix}$$

$$\begin{array}{l} r_1 \\ r_2 + r_1 \\ r_3 - 2r_1 \end{array} \quad \begin{bmatrix} 1 & -2 & 3 & 5 \\ 0 & 0 & 0 & 9 \\ 0 & 5 & -7 & -6 \end{bmatrix} \quad \begin{array}{l} \\ \leftarrow \text{new } r_2 \\ \leftarrow \text{new } r_3 \end{array}$$

Row 2 is suggesting that $0x + 0y + 0z = 9$.

No values for x, y and z can satisfy this equation so again we conclude that the sysyem has no solution.

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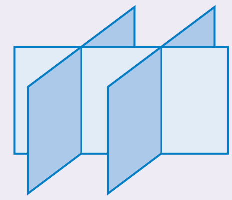
EXAMPLE 3 REPEATED — Solution without using augmented matrix approach

Determine whether the system shown on the right has a unique solution, no solution or an infinite number of solutions. If there is a unique solution, find it.

$$\begin{aligned} x - 2y + 3z &= 5 \\ -x + 2y - 3z &= 4 \\ 2x + y - z &= 4 \end{aligned}$$

Solution

As before, noticing that the first two equations define separate parallel planes and the third equation is not parallel to the other two, we have the no solution situation shown on the right.



Alternatively, had the parallel nature of two of the planes escaped our notice, manipulation of the equations would yield the same conclusion:

Given the three equations:

$$\begin{cases} x - 2y + 3z = 5 & [1] \\ -x + 2y - 3z = 4 & [2] \\ 2x + y - z = 4 & [3] \end{cases}$$

Use equations [1] and [2] to eliminate x.

Equation [1] $\quad x - 2y + 3z = 5$

Equation [2] $\quad -x + 2y - 3z = 4$

Addition eliminates x (and y and z!!) $\quad 0x + 0y + 0z = 9 \quad$ i.e. $0 = 9$!!

No values for x, y and z can satisfy this equation so again we conclude that the system of equations has no solution.

How does your calculator respond when asked to solve a system of equations for which there is no solution?
How does it respond when asked to solve systems for which there are infinite solutions?

EXAMPLE 4 — Augmented matrix approach

For what value(s) of p will the system of equations shown on the right have a unique solution?

$$\begin{aligned} x - 2y + z &= -3 \\ -x + 3y + z &= -2 \\ 3x - 5y + pz &= 7 \end{aligned}$$

Solution

The augmented matrix for the given system is:

$$\begin{array}{l} r_1 \\ r_2 \\ r_3 \end{array} \left[\begin{array}{cccc} 1 & -2 & 1 & -3 \\ -1 & 3 & 1 & -2 \\ 3 & -5 & p & 7 \end{array}\right]$$

$$\begin{array}{l} r_1 \\ r_2 + r_1 \\ r_3 - 3r_1 \end{array} \left[\begin{array}{cccc} 1 & -2 & 1 & -3 \\ 0 & 1 & 2 & -5 \\ 0 & 1 & p-3 & 16 \end{array}\right] \begin{array}{l} \\ \leftarrow \text{new } r_2 \\ \leftarrow \text{new } r_3 \end{array}$$

$$\begin{array}{l} r_1 \\ r_2 \\ r_3 - r_2 \end{array} \left[\begin{array}{cccc} 1 & -2 & 1 & -3 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & p-5 & 21 \end{array}\right] \begin{array}{l} \\ \leftarrow \text{new } r_2 \\ \leftarrow \text{new } r_3 \end{array}$$

Thus a unique solution exists provided $p - 5 \neq 0$, i.e provided $p \neq 5$.

EXAMPLE 4 REPEATED **Solution without using augmented matrix approach**

For what value(s) of p will the system of equations shown on the right have a unique solution?

$$\begin{aligned} x - 2y + z &= -3 \\ -x + 3y + z &= -2 \\ 3x - 5y + pz &= 7 \end{aligned}$$

Solution

Given the three equations: $\begin{cases} x - 2y + z = -3 & [1] \\ -x + 3y + z = -2 & [2] \\ 3x - 5y + pz = 7 & [3] \end{cases}$

Use equations [1] and [2] to eliminate x.

Equation [1]	$x - 2y + z = -3$	
Equation [2]	$-x + 3y + z = -2$	
Addition eliminates x.	$y + 2z = -5$	[4]

Use equations [1] and [3] to eliminate x.

Equation [1] $\times$ 3	$3x - 6y + 3z = -9$	
Equation [3]	$3x - 5y + pz = 7$	
Subtraction eliminates x.	$-y + (3 - p)z = -16$	[5]

Use equations [4] and [5] to eliminate y.

Equation [4]	$y + 2z = -5$
Equation [5]	$-y + (3 - p)z = -16$
Addition eliminates y.	$(5 - p)z = -21$

This last equation gives a solution provided $5 - p \neq 0$, otherwise $0z = -21$!

A unique solution exists provided $5 - p \neq 0$, i.e provided $p \neq 5$.

Alternatively, with no parallel or coincident planes involved, example 4 could be solved by considering linear combinations, as shown below. However, it is anticipated that for most students, eliminating the variables by one or other of the methods of presentation just demonstrated would be the preferred method of solution for questions of this type.

- If there exists a linear combination of the first two equations, equal to the third equation, we really only have two pieces of information about three unknowns. Hence we would have infinite solutions.
- If there exists a linear combination of the first two equations that differs from the third equation only in the last column we have contradictory statements. Hence we would have no solution.

Consider $4 \times [1] + 1 \times [2]$:
(The 4 and the 1 being obtained by solving $1\lambda - 1\mu = 3$ with $-2\lambda + 3\mu = -5$.)

$$\begin{aligned} 4(x - 2y + z &= -3) + \\ 1(-x + 3y + z &= -2) \\ \hline 3x - 5y + 5z &= -14 \end{aligned}$$

Compare this with [3]:

$$3x - 5y + pz = 7$$

If $p = 5$ we have a contradiction: $3x - 5y + 5z = -14$
and $3x - 5y + 5z = 7$

This is avoided, and a unique solution will be obtained, if $p \neq 5$.

Thus, as before, a unique solution exists provided $p \neq 5$.

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EXAMPLE 5 Augmented matrix approach

Determine the possible values of p and q if the system of equations shown on the right has

$$\begin{aligned} x - y + 2z &= 1 \\ 2x - 5y + 5z &= 9 \\ 3x + 3y + pz &= q \end{aligned}$$

a no solution,

b infinite solutions,

c a unique solution.

Solution

Using the augmented matrix:

$$\begin{array}{l} r_1 \\ r_2 \\ r_3 \end{array} \quad \begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & -5 & 5 & 9 \\ 3 & 3 & p & q \end{bmatrix}$$

$$\begin{array}{l} r_1 \\ r_2 - 2r_1 \\ r_3 - 3r_1 \end{array} \quad \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & -3 & 1 & 7 \\ 0 & 6 & p-6 & q-3 \end{bmatrix} \quad \begin{array}{l} \\ \leftarrow \text{new } r_2 \\ \leftarrow \text{new } r_3 \end{array}$$

$$\begin{array}{l} r_1 \\ r_2 \\ r_3 + 2r_2 \end{array} \quad \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & -3 & 1 & 7 \\ 0 & 0 & p-4 & q+11 \end{bmatrix} \quad \begin{array}{l} \\ \\ \leftarrow \text{new } r_3 \end{array}$$

Thus the system has

a no solution if $p = 4$ and $q \neq -11$ (as we then have $0x + 0y + 0z =$ non zero)

b infinite solutions if $p = 4$ and $q = -11$ (as we then have $0x + 0y + 0z = 0$)

c a unique solution if $p \neq 4$.

Again these same answers could be obtained by considering what linear combination of the first two equations could give the third equation:

Solving $1\lambda + 2\mu = 3$ with $-1\lambda - 5\mu = 3$ gives $\lambda = 7$ and $\mu = -2$.

Thus we try (7) (1st row) + (–2) (2nd row):

$$\begin{array}{rrrr} (7)\,(1 & -1 & 2 & 1)\ + \\ (-2)\,(2 & -5 & 5 & 9) \\ \hline 3 & 3 & 4 & -11 \end{array}$$

Comparing this with the third row: $\quad 3 \quad 3 \quad p \quad q$

we again conclude that the system has

a no solution if $p = 4$ and $q \neq -11$, as then we would have contradictory equations. (Inconsistent.)

b infinite solutions if $p = 4$ and $q = -11$, as then we have a repeat equation. (Insufficient.)

c a unique solution if $p \neq 4$.

EXAMPLE 5 REPEATED Solution without using augmented matrix

Determine the possible values of p and q if the system of equations shown on the right has

$$\begin{aligned} x - y + 2z &= 1 \\ 2x - 5y + 5z &= 9 \\ 3x + 3y + pz &= q \end{aligned}$$

a no solution,

b infinite solutions,

c a unique solution.

Solution

Given the three equations: $\begin{cases} x - y + 2z = 1 & [1] \\ 2x - 5y + 5z = 9 & [2] \\ 3x + 3y + pz = q & [3] \end{cases}$

Use equations [1] and [2] to eliminate x.

Equation [1] $\times$ 2 $\quad 2x - 2y + 4z = 2$
Equation [2] $\quad 2x - 5y + 5z = 9$

Subtraction eliminates x. $\quad 3y - z = -7 \quad [4]$

Use equations [1] and [3] to eliminate x.

Equation [1] $\times$ 3 $\quad 3x - 3y + 6z = 3$
Equation [3] $\quad 3x + 3y + pz = q$

Subtraction eliminates x. $\quad -6y + (6 - p)z = 3 - q \quad [5]$

Use equations [4] and [5] to eliminate y.

Equation [4] $\times$ 2 $\quad 6y - 2z = -14$
Equation [5] $\quad -6y + (6 - p)z = 3 - q$

Addition eliminates y. $\quad (4 - p)z = -11 - q$

If $p = 4$ and $q \neq -11$ we have $0z \neq 0$. No solution.

If $p = 4$ and $q = -11$ we have $0z = 0$. Infinite solutions.

If $p \neq 4$ we have 'non zero z equalling some number'. Unique solution.

Thus

a no solution if $p = 4$ and $q \neq -11$,

b infinite solutions if $p = 4$ and $q = -11$,

c a unique solution if $p \neq 4$.

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Exercise 6B

Determine the value of k in each of the following systems of equations given that each system has no solution.

1
$$\begin{aligned} x + 2y + z &= 3 \\ y + 4z &= 1 \\ kz &= 5 \end{aligned}$$

2
$$\begin{aligned} x + 3y + 2z &= 4 \\ -y + 3z &= 1 \\ (k-2)z &= 3 \end{aligned}$$

3
$$\begin{aligned} x - 2y + z &= 4 \\ y + 3z &= 1 \\ (2k+1)z &= 2 \end{aligned}$$

4
$$\begin{aligned} x + 3y + 2z &= 1 \\ y + kz &= 2 \\ -y + 3z &= 5 \end{aligned}$$

5
$$\begin{aligned} 2x + y + 4z &= 2 \\ 3y + kz &= 4 \\ 2y + z &= 3 \end{aligned}$$

6
$$\begin{aligned} x + 2y + z &= 3 \\ y - 3z &= k \\ -2y + 6z &= -4 \end{aligned}$$

7
$$\begin{aligned} (k-2)z &= 4 \\ y + 2z &= -3 \\ x + 2y + 3z &= 5 \end{aligned}$$

8
$$\begin{aligned} y + 3z &= 2 \\ (k+1)z &= 5 \\ x + 3y - z &= 2 \end{aligned}$$

9
$$\begin{aligned} x + 2y + kz &= 1 \\ 2x - 3y + z &= 5 \\ 3x - y + 4z &= 3 \end{aligned}$$

10
$$\begin{aligned} x + 3y - 6z &= 3 \\ x + y + z &= 0 \\ 3x + 5y + (k+1)z &= 2 \end{aligned}$$

11
$$\begin{aligned} 2x + y + kz &= -1 \\ x + 4y + 2z &= -7 \\ 3x - 2y + 4z &= 1 \end{aligned}$$

12
$$\begin{aligned} x + y + 3z &= 4 \\ -x + 5y + (k+1)z &= 6 \\ 2x - y + z &= 5 \end{aligned}$$

Determine the value of k in each of the following systems of equations given that each system has infinite solutions.

13
$$\begin{aligned} x + 3y - 2z &= 5 \\ y - 2z &= 4 \\ kz &= 0 \end{aligned}$$

14
$$\begin{aligned} x + 2y - z &= 1 \\ z &= 2 \\ (2k+1)z &= 0 \end{aligned}$$

15
$$\begin{aligned} x - 3y + 5z &= 5 \\ -y + 2z &= 8 \\ (k^2-4)z &= k+2 \end{aligned}$$

16
$$\begin{aligned} 2kz &= 0 \\ 3y + 2z &= 5 \\ x - y + 3z &= 5 \end{aligned}$$

17
$$\begin{aligned} 2y + z &= 2 \\ ky &= 0 \\ x + 3y + 4z &= 2 \end{aligned}$$

18
$$\begin{aligned} x - 2y + 3z &= -1 \\ 2x - 4y + 6z &= -2 \\ -x + 2y + kz &= 1 \end{aligned}$$

19
$$\begin{aligned} x - y + z &= 3 \\ 2x + 3y + kz &= 2 \\ 4x + 11y - 5z &= 0 \end{aligned}$$

20
$$\begin{aligned} x + 3y - 2z &= 4 \\ x + 5y + (k-2)z &= 3 \\ 2x + (k+1)y - 7z &= 9 \end{aligned}$$

21 For the system of equations $\begin{cases} x + py = 5 \\ 2x + 3y = q \end{cases}$ determine the values of p and q

a for the system to have infinite solutions,

b for the system to have no solution,

c for the system to have a unique solution.

22 For the system of equations $\begin{cases} px + 4y = 6 \\ 9x + 6y = q \end{cases}$ determine the values of p and q

a for the system to have infinite solutions,

b for the system to have no solution,

c for the system to have a unique solution.

For questions **23** and **24** determine the values of p and q for which the system of equations has infinite solutions.

23
$$\begin{aligned} x + 2y + z &= 3 \\ x + 3y - 2z &= 7 \\ 3x + 4y + pz &= q \end{aligned}$$

24
$$\begin{aligned} x + 3y - z &= 2 \\ 2x + 8y - 2z &= q \\ x - 3y + pz &= -1 \end{aligned}$$

For questions **25** and **26** determine the value(s) of p for which the system of equations has a unique solution.

23
$$\begin{aligned} x - 2y + z &= -2 \\ x + y + z &= 7 \\ -x + 5y + pz &= -4 \end{aligned}$$

24
$$\begin{aligned} 5x + 2y - z &= 2 \\ 2x + y &= 1 \\ -3x + y + pz &= 1 \end{aligned}$$

27 Determine whether the system shown on the right has a unique solution, no solution or infinite solutions and, if there is a unique solution, determine that solution.

$$\begin{aligned} x + 3y - z &= 5 \\ -x + 3y + z &= 5 \\ 2x + 6y - 2z &= 10 \end{aligned}$$

28 State the possible values of k and m if the system of equations shown below has

a a unique solution, **b** no solution, **c** infinite solutions.

$$x + 2y + z = 4, \qquad y - 3z = 1, \qquad (2k - 1)y = m + 1.$$

29 Determine whether the system shown on the right has a unique solution, no solution or infinite solutions for each of the following cases and, if there is a unique solution, determine that solution.

$$\begin{aligned} x - y + 2z &= 12 \\ -x - 2y + z &= 3 \\ 8x + 7y + pz &= q \end{aligned}$$

a $p = 1, q = 10$, **b** $p = 1, q = 21$, **c** $p = 7, q = 45$.

30 Determine the possible values of k and m if the system on the right has

$$\begin{aligned} x - y &= m \\ x + ky - 3z &= 7 \\ 4x - y - 3z &= 3 \end{aligned}$$

a a unique solution,

b no solution,

c infinite solutions.

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Miscellaneous exercise six

This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters, and the ideas mentioned in the Preliminary work section at the beginning of the book.

1 Use vectors to prove that the medians of a triangle intersect at a point two-thirds of the way along each median, measured from the end of the median that is a vertex of the triangle.
(A median of a triangle is a straight line drawn from a vertex of the triangle to the midpoint of the opposite side.)

2 **a** Given that $|x - a| = 5 - |x + 3|$ has exactly two solutions find the range of values a can take.

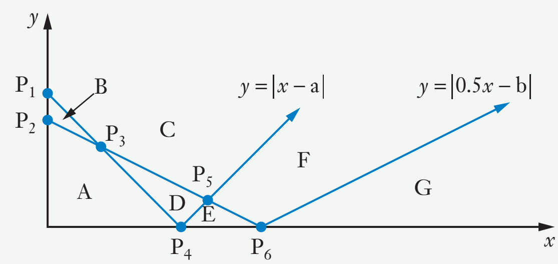
b Given that $|x - a| = 5 - |x + 3|$ has more than two solutions find the range of values a can take.

3 The diagram on the right shows

$$y = |x - a| \quad \text{for } x \geq 0 \text{ and } y \geq 0,$$

and $$y = |0.5x - b| \quad \text{for } x \geq 0 \text{ and } y \geq 0,$$

dividing the graph into seven regions (labelled A to G in the diagram).



a Find the coordinates of P_1 and P_2, in terms of a and/or b.

b Is $a > b$, or is $b > a$?

c Find the coordinates of P_4 and P_6 (in terms of a and/or b).

d Find the coordinates of P_3 and P_5 (in terms of a and/or b).

State the regions A, B, C, etc. which together form each of the following sets of points for $x \geq 0$ and $y \geq 0$:

e $\{(x, y): y < |x - a| \text{ and } y < |0.5x - b|\}$

f $\{(x, y): y > |x - a| \text{ and } y > |0.5x - b|\}$

g $\{(x, y): y < |x - a| \text{ and } y > |0.5x - b|\}$

h $\{(x, y): y > |x - a| \text{ and } y < |0.5x - b|\}$

4 Represent the set of points $\{z: |z - 3 - i| \leq 3\}$ as a shaded region on an Argand diagram.

5 Express **a** $-5(\sqrt{3} + i)$ in the form $r\,\text{cis}\,\theta$ with $r \geq 0$ and $-\pi < \theta \leq \pi$,

b $6\,\text{cis}\left(\frac{3\pi}{4}\right)$ in the form $a + bi$.

6 With $z = x + iy$ and $3|z - 5| = 2|z + 5i|$ prove that

$$(x - 9)^2 + (y - 4)^2 = 72.$$

7 If $z = 1 - i$ **a** express z in the form $r\,\text{cis}\,\theta$ with $-\pi < \theta \leq \pi$,

b express z^{14} in cartesian form.

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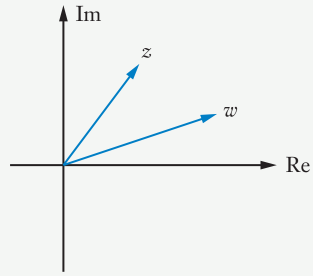
8 Find the position vector of the point where lines L_1 and L_2 intersect where

L_1 has equation $\quad \mathbf{r} = 2\mathbf{i} + 2\mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k})$

and L_2 has equation $\quad \mathbf{r} = -2\mathbf{i} + \mathbf{j} + 6\mathbf{k} + \mu(-2\mathbf{i} - \mathbf{j} + 2\mathbf{k})$.

9 The diagram on the right shows the complex numbers z and w as vectors on an Argand diagram.



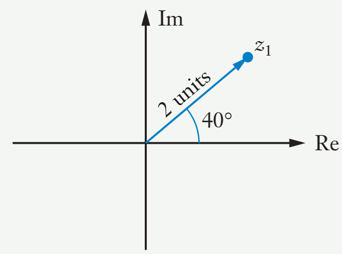
a Make a copy of the diagram and include on your diagram $\bar{z}$ and $\bar{w}$, the complex conjugates of z and w.

b With the aid of a diagram, prove that

$$\overline{z + w} = \bar{z} + \bar{w}$$

c Similarly use a diagrammatic approach to show that $\overline{zw} = \bar{z}\,\bar{w}$.

10 z_1 shown in the diagram on the right is one solution to the equation $z^4 = \text{k}$.



Find k and z_2, z_3 and z_4, the other three solutions to the equation, giving all answers in the form $r \text{ cis } \theta°$ for $r \geq 0$ and $-180 < \theta \leq 180$.

11 **a** State the natural domain, and corresponding range of the function

$$f(x) = \frac{1}{\sqrt{x - 3}} + 4$$

b Find an expression for $f^{-1}(x)$, the inverse of $f(x)$ and state the domain and range of $f^{-1}(x)$.

12 Use de Moivre's theorem to express $\cos 4\theta$ and $\sin 4\theta$ in terms of $\cos\theta$ and $\sin\theta$.

13 Two planes have equations $\quad \mathbf{r} . \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} = -14 \quad$ and $\quad \mathbf{r} . \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} = 42.$

Prove that the two planes are parallel and find the distance between them.

14 Find the shortest distance from the line $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}$ to the origin.

15 Prove that the system of equations given below has either no solution or infinite solutions and find the value of p that gives infinite solutions.

$$\text{p}z - 3y = 1 + x \qquad x + 3y = 3 + 2z \qquad 2x + 6y - 6z = 5$$

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